Hall’s problem on conciseness of words

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Congrats to Evgeny!
Let $w = w(x_1, \ldots, x_k)$ be a group-word, and let $G$ be a group. The verbal subgroup $w(G)$ of $G$ determined by $w$ is the subgroup generated by the set $G_w$ consisting of all values $w(g_1, \ldots, g_k)$, where $g_1, \ldots, g_k$ are elements of $G$. A word $w$ is said to be concise if whenever $G_w$ is finite for a group $G$, it always follows that $w(G)$ is finite. More generally, a word $w$ is said to be concise in a class of groups $X$ if whenever $G_w$ is finite for a group $G \in X$, it always follows that $w(G)$ is finite. In the sixties P. Hall asked whether every word is concise. The negative answer to Hall’s problem was given by Ivanov in 1989.
On the other hand, many important words are known to be concise. In some cases proving conciseness is quite easy.

Indeed, suppose that $w$ is a word having only $m$ values in the group $G$. Set $M = w(G)$. It is clear that $C_G(M)$ has finite $m$-bounded index. Therefore the index $[M : Z(M)]$ is finite and $m$-bounded. So by Schur’s theorem the commutator subgroup $M'$ is of finite order. Factoring it out we can assume that the verbal subgroup $M$ is abelian. In the case where $w = x^k$ we see that $M^k$ has order at most $m$ and so $M$ is an abelian group with small number of generators and small exponent. Hence, $M$ is finite (of bounded order).
In the sixties and seventies a number of deeper results on conciseness of words were obtained.
It was shown by Jeremiah Wilson that the multilinear commutator words are concise. Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials. Merzlyakov showed that every word is concise in the class of linear groups while Turner-Smith proved that every word is concise in the class of residually finite groups all of whose quotients are again residually finite.
The negative solution of Hall’s problem was obtained by Ivanov in 1989 by constructing a group $G$ admitting a word $w$ that takes precisely two values in $G$ and has $w(G)$ infinite cyclic. The group $G$ constructed by Ivanov is not residually finite. Recently, Hall’s problem for residually finite groups was mentioned by Jaikin-Zapirain and Segal. (Actually Jaikin-Zapirain formulated it for profinite groups). In a sense, the residually finite case of the problem is more interesting than the original version, since it allows use of a greater variety of tools. In particular, the results obtained in the context of the restricted Burnside problem seem of relevance here.
How the RBP is related to the Hall problem for residually finite groups? I would like to illustrate this using the example of the $n$th Engel word $e_n = [x, y, \ldots, y]$, where $y$ is repeated $n$ times. It is unknown whether these words are concise. Fernández- Alcober, Morigi and Traustason and, independently, Abdollahi and Russo showed in 2011 that these words are concise if $n \leq 4$. We will now sketch out the proof that the word $e_n$ is concise in the class of residually finite groups for every $n$. 
Thus, let $n \geq 1$ and $G$ be a residually finite group with precisely $m$ values of the word $w = e_n$. Without loss of generality we can assume that $G$ is generated by at most $2m$ elements and the verbal subgroup $M = w(G)$ is abelian. It follows that if $x \in M$ and $y \in G$, for each $i$ we have $[x^i, y, \ldots, y] = [x, y, \ldots, y]^i$. This shows that every power of such a $w$-value $[x, y, \ldots, y]$ is again a $w$-value. Therefore any cyclic subgroup generated by an element of the form $[x, y, \ldots, y]$ with $x \in M$ has order dividing $m$. 

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Recall that $M$ is an abelian $m$-generated group. It is clear that the subgroups of $M$ of order dividing $m$ generate a subgroup of order at most $m^m$. We can factor out this subgroup and assume that $[M, y, \ldots, y] = 1$ for any $y \in G$. Since for each $x \in G$ the element $[x, y, \ldots, y] \in M$, we now conclude that $[x, y, \ldots, y] = 1$ for all $x, y \in G$. Thus, our group $G$ is now $2n$-Engel.

It now follows from Zelmanov’s results on Engel groups that $G$ is nilpotent with $(m, n)$-bounded nilpotency class.
So our group $G$ is now nilpotent of $m$-bounded class and has $m$-bounded number of generators. Every word in such a group has $m$-bounded width. That is, every element in $w(G)$ is a product of $m$-boundedly many $w$-values (or their inverses). Since $G$ has only $m$ $w$-values, we conclude that $w(G)$ is finite with bounded order.

This proves that if a residually finite group $G$ has only $m$ values of the $n$th Engel word, then the corresponding verbal subgroup in $G$ is finite with $(m, n)$-bounded order.
With appropriate modifications these arguments lead to similar conclusions about many other words. Using a combination of results of Burns, Medvedev and Groves we deduce that the word $w$ is concise in residually finite groups whenever for any $n$ the word $w$ is not a law in the wreath product $C_n \wr C$. Here $C_n$ is cyclic of order $n$ and $C$ is infinite cyclic. In particular, any word of the form $uv^{-1}$, where $u$ and $v$ are positive words, is concise in residually finite groups. Other examples include generalizations of the Engel word like $[x^{n_1}, y^{n_2}, \ldots, y^{n_k}]$ etc. These are the words $w$ that imply virtual nilpotence of finitely generated metabelian groups satisfying the law $w \equiv 1$. 

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Let us say that a word $w$ is boundedly concise in a class of groups $X$ if for every integer $m$ there exists a number $\nu = \nu(X, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in X$ it always follows that $|w(G)| \leq \nu$. Fernández-Alcober and Morigi showed that every word which is concise in the class of all groups is actually boundedly concise. It is unclear if every word which is concise in the class of residually finite groups is actually boundedly concise. We just have shown that the $n$th Engel word is boundedly concise in residually finite groups. In fact, in all cases where a word is known to be concise in residually finite groups, we were able to prove that it is boundedly concise. Therefore we conjecture that

Any group-word that is concise in the class of residually finite groups is in fact boundedly concise.
Essentially, this is a problem on finite groups. Since we still hope that every word is concise in the class of residually finite groups, we formulate the problem as follows:

Let $w$ be a word and $G$ a finite group in which $w$ takes only $m$ values. Is the order of $w(G)$ bounded in terms of $m$ and $w$?

Thus, the original problem of Hall “evolved” and became a problem on finite groups.
The natural candidates for the negative solution of our problem are the words that are not concise in the class of all groups. For example, the word from Ivanov’s work

$$[[x^{pn}, y^{pn}]^n, y^{pn}]^n,$$

where $n$ is odd $\geq 10^{10}$ and $p$ is a prime $\geq 5000$. However it seems hard to work with this word. Perhaps the shortest word about which we cannot say whether or not it is concise in residually finite groups is

$$[x^k, y^l]^n,$$

where $n$ is not a prime-power. We will now talk about cases of positive solutions for the problem.
Recall that a multilinear commutator word (outer commutator word) is a word which is obtained by nesting commutators, but using always different variables. Thus the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator while the Engel word $[x_1, x_2, x_2, x_2]$ is not. An important family of multilinear commutator words is formed by the derived words $\delta_k$, on $2^k$ variables, which are defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$, the $k$-th derived subgroup of $G$. Another distinguished family of multilinear commutators are the simple commutators $\gamma_k$, given by

$$\gamma_1 = x_1, \quad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k].$$

The corresponding verbal subgroups $\gamma_k(G)$ are the terms of the lower central series of $G$. 
In 2014 in a joint work with Acciarri we showed that

Whenever $q$ is a prime-power, the word $\gamma^q_k$ is boundedly concise in the class of residually finite groups.

It is unknown whether this word is concise (in the class of all groups). The proof uses the techniques employed in the solution of the RBP.
In 2015 in a joint work with Guralnick we showed that the word \( w = [\ldots [x_1^{n_1}, x_2]^{n_2}, \ldots, x_k]^{n_k} \) is boundedly concise in the class of residually finite groups for any integers \( n_1, n_2, \ldots, n_k \).

In particular, it follows that the word \( \gamma_k^n \) is boundedly concise in the class of residually finite groups for any \( n \).

It is unknown whether these words are concise (in the class of all groups). The proofs of the above results were obtained using character theory.
Another result in this direction was obtained in a joint work with Fernández-Alcober.

If \( q \) is a prime-power and \( w \) is a multilinear commutator, then the word \( w^q \) is boundedly concise in residually finite groups.

The proof is based on Lie-theoretical techniques employed in the solution of the RBP.
I would like to describe some ideas used in the work with Guralnick. We will illustrate that work using the example of the word $w = [x, y]^n$. It is unknown whether this word is concise (in the class of all groups). We will show that it is boundedly concise in the class of residually finite groups for any $n$. 
We require the following elementary lemma.

Lemma: Let $G$ be a finite group and $x, y \in G$. Then we have $\langle x \rangle = \langle y \rangle$ if and only if $y = x^e$ for some $e$ such that $(e, |G|) = 1$. 
We will use the property that if \( G \) is a finite group and \( x \in G \) is a commutator, then every element \( y \in \langle x \rangle \) such that \(|y| = |x|\) is again a commutator.

Now fix \( n \) and let \( w = [x, y]^n \). Let \( G \) be a finite group in which \( w \) takes only \( m \) \( w \)-values. We wish to show that \( w(G) \) has \( m \)-bounded order. As before, we can assume that \( w(G) \) is abelian. Thus, this is an abelian \( m \)-generated subgroup. It is sufficient to bound the order of every \( w \)-value – then a bound on the order of \( w(G) \) will follow.
We know that the number of generators of a cyclic group of order $t$ is precisely $\phi(t)$ – the Euler function. Of course, this number tends to $\infty$ when $t$ does.

Now let $a$ be a $w$-value in $G$, that is, $a = [x, y]^n$ for suitable $x, y$. Suppose that $a$ is of order $t$ and set $s = \phi(t)$. The cyclic subgroup $\langle a \rangle$ has $s$ elements $a_1, \ldots, a_s$ such that $\langle a_i \rangle = \langle a \rangle$. For each $a_i$ there exists a number $e_i$ with the properties that $a^{e_i} = a_i$ and $(e_i, |G|) = 1$. The latter property guarantees that $[x, y]^{e_i}$ is again a commutator. So we have

$$a_i = a^{e_i} = ([x, y]^n)^{e_i} = ([x, y]^{e_i})^n.$$ 

This shows that each element $a_i$ is a $w$-value. Therefore $s \leq m$ and $t$, the order of $a$, is $m$-bounded. Hence the word $w = [x, y]^n$ is boundedly concise in the class of residually finite groups for any $n$. 

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The main novelty of the paper with Guralnick was the discovery that all words of the form

\[ w = [\ldots [x_1^{n_1}, x_2]^{n_2}, \ldots, x_k]^{n_k} \]

have the property that if \( g \) is a \( w \)-value in a finite group and \( h \in \langle g \rangle \) is such that \( |h| = |g| \), then also \( h \) is a \( w \)-value.
Now about the word $w = [u, [x, y], \ldots, [x, y]]$, where $[x, y]$ is repeated $n$ times. Whenever $G$ satisfies the law $w \equiv 1$, all commutators in $G$ are $n$-Engel.

Suppose $G$ is a finite group in which $w$ takes at most $m$ values. As before, we can assume that $G$ has small number of generators and $M = w(G)$ is abelian. For each $i$ and $h \in M$ we have $[h^i, [x, y], \ldots, [x, y]] = [h, [x, y], \ldots, [x, y]]^i$. Therefore we can assume that

$$[M, \overbrace{[x, y], \ldots, [x, y]}^n] = 1$$

for any $x, y \in G$. So each commutator acts on $M$ as an $n$-Engel element. Thus, each commutator in $2n$-Engel in $G$. 

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Since $G$ is generated with bounded number of elements, it follows that each element of $G'$ is a product of boundedly many commutators. Using Zelmanov’s theory we can now deduce that $G'$ is $n_1$-Engel for some bounded number $n_1$. A result of Burns and Medvedev now tells us that $G'$ has a characteristic subgroup $E$ of bounded exponent such that $G'/E$ is nilpotent with bounded class. (The proof of this is also based on the RBP.) Since $M$ is abelian of small rank, the intersection $M \cap E$ has small order and so we can pass to the quotient $G/E$. Thus, we can assume that $G'$ is nilpotent with bounded nilpotency class.
Now what? Solving the Hall problem for abelian-by-nilpotent groups is actually not so difficult. But our group is nilpotent-by-abelian and there is no general method for solving the Hall problem for such groups. Using induction on the class of $G'$ we assume that there exists a bounded number $f$ such that $M^f$ is contained in the last term of the lower central series of $G'$. 
We will use the formula \([b, a, a] = [a^{-ba}, a]\). Choose \(x, y, u \in G\) such that \([u, [x, y], \ldots, [x, y]] \neq 1\) where \([x, y]\) is repeated \(n\) times. Set \(a = [x, y]\) and \(b = [x, y]^{-u}\). Thus, the element \([b, a, \ldots, a]_{n-1}\) represents a nontrivial \(w\)-value. Choose a positive integer \(l\) which is coprime with the order of \(a\). We know that \(a^l\) is a commutator. It is clear that \(b^l\) is a commutator conjugate to \(a^{-l}\) and we deduce that the element \([b^l, a^l, \ldots, a^l]_{n-1}\) is a \(w\)-value.
Now a more technical part: If $G'$ is of class $c$ then

$$\left[ b^l, a^l, \ldots, a^l \right]^f = \left[ b, a, \ldots, a \right]^{fl^c}_{n-1}. \quad \left[ b, a, \ldots, a \right]^{fl^c}_{n-1}$$

So we have a formula that relates powers of $w$-values. With more work we can now force $w$-values to have small order. Precisely as required.
That is it.

Thank you!