Generator-Inverting Group Automorphisms

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Definition of GI-Automorphism

Let $G$ be a group. Suppose $\sigma \in \text{Aut}(G)$ satisfies $\sigma^2 = 1$. Let $X = \{ g \in G : \sigma(g) = g^{-1} \}$. Call $\sigma$ a **GI-automorphism** of $G$ if $\langle X \rangle = G$.

**Trivial Examples:**
1. If $G$ is generated by elements of order 2, then $1 \in \text{Aut}(G)$ is a GI-automorphism.
2. If $G$ is abelian, then $x \mapsto x^{-1}$ defines a GI-automorphism.
Suppose $\sigma \in \text{Aut}(G)$ is a GI-automorphism. Then $H := G \rtimes \langle \sigma \rangle$ defines the corresponding **GI-extension**.

Equivalently, a GI-extension of $G$ is a group containing $G$ with index 2 and generated by involutions outside $G$.

**Examples:**

(1) If $\sigma = 1 \in \text{Aut}(G)$ is a GI-automorphism, then the corresponding GI-extension $\cong G \times C_2$.

(2) If $G$ is abelian and $\sigma : x \mapsto x^{-1}$, then the GI-extension is $\text{Dih}(G)$.

(3) $A_5$ has 2 GI-extensions, $A_5 \times C_2$ and $S_5$. 
(1) Let \( M \) be a closed orientable irreducible 3-manifold and \( G = \pi_1(M) \). Thurston asked if \( G \) has a GI-automorphism.

**Results:**

If \( M \) is hyperbolic and \( G \) is 2-generated, then the answer is yes by Jorgensen’s construction (if \( G = \langle A, B \rangle \), then let \( \sigma \) be conjugation by \( AB - BA \in PGL(2, \mathbb{C}) \)).

Boileau-Weidmann (2008): if \( M \) is a graph manifold and \( G \) is 2-generated, then \( G \) has a GI-automorphism if and only if \( M \) has Heegaard genus 2.
(2) Let $K$ be a quadratic number field and $L/K$ be an unramified extension, Galois over $\mathbb{Q}$. Then the inertia subgroups of $\text{Gal}(L/\mathbb{Q})$ all have order 2, lie outside $\text{Gal}(L/K)$, and by Minkowski generate $\text{Gal}(L/\mathbb{Q})$.

Thus $\text{Gal}(L/\mathbb{Q})$ is a GI-extension of $\text{Gal}(L/K)$.

(3) Groups on Wil Cocke’s poster have no GI-automorphism.
First Results

(1) The smallest groups with no GI-automorphism are the Frobenius groups of order 20 and 21.

(2) The Frobenius group with kernel \(\mathbb{F}_q^+\), of characteristic \(p\), and complement cyclic of order \(d\) has a GI-automorphism if and only if there exists \(\ell\) such that \(p^\ell \equiv -1 \pmod{d}\) (Alberts).

(3) There are infinitely many 2-groups with no GI-automorphism (B, Leedham-Green).

(4) If \(G\) is a 2-generated 2-group, then it has at most one GI-extension.

(5) Since “most” \(p\)-groups have automorphism group a \(p\)-group, if \(p\) is odd, these groups do not have a GI-automorphism.
Suppose $p$ is an odd prime.
Let $c_p = \prod_{n \geq 1} (1 - p^{-n})$.
If $G$ is an abelian $p$-group, set $\text{Meas}_{\text{CL}}(G) := c_p / |\text{Aut}(G)|$.
This defines a probability measure on the set of abelian $p$-groups.

Cohen and Lenstra (1983) conjectured that the proportion of imaginary quadratic $K$ whose maximum unramified abelian $p$-extension has Galois group $G$, is $\text{Meas}_{\text{CL}}(G)$.
Suppose $p$ is an odd prime and $K$ imaginary quadratic. The Galois group $G$ of the maximum unramified $p$-extension (its $p$-tower group) is a (possibly infinite) pro-$p$ group satisfying

1. $G$ has a GI-automorphism;
2. $d(G) = r(G)$;
3. $G$ has finite abelianization.

These properties define a **Schur $\sigma$-group**.

The $p$-class $c$ quotient of $G$, for any $c$ and any Schur $\sigma$-group $G$, is called a **Schur $\sigma$-quotient**.
Suppose $G$ has a GL-automorphism $\sigma$.
Let $\text{Aut}_\sigma(G)$ denote the group of $\sigma$-equivariant automorphisms of $G$, i.e. the centralizer in $\text{Aut}(G)$ of $\sigma$.
Fix a finite Schur $\sigma$-group $G$.
Set $\text{Meas}_{BBH}(G) := c_p/|\text{Aut}_\sigma(G)|$.
$\text{Meas}_{BBH}$ defines a non-discrete measure on certain pro-$p$ groups.

B-Bush-Hajir conjecture that the proportion of imaginary quadratic $K$ whose $p$-tower group is isomorphic to $G$, is $\text{Meas}_{BBH}(G)$. 


Meas_{CL}(G) is the proportion of \( d \)-tuples of relators, taken from the free abelian pro-\( p \) group of rank \( d \), that present \( G \).

Meas_{BBH}(G) is the proportion of \( d \)-tuples of relators, inverted by \( \sigma \) and taken from the free pro-\( p \) group of rank \( d \), that present \( G \).

Likewise, given a \( p \)-group \( G \) of \( p \)-class \( c \), we can ask for the proportion of \( d \)-tuples taken from a free pro-\( p \) group of \( p \)-class \( c \) and rank \( d \) that are inverted by \( \sigma \) and present \( G \).

This yields measures on Schur \( \sigma \)-quotients. The following (partial) tree gives some for \( d = 2 \). The top group has order 27 and 3-class 2 and each group connects down to its Schur children (descendants of 3-class one more).
Moments

There is data in support of these heuristics, but other results too. If $H$ is a finite $p$-group having a GI-automorphism $\sigma$, then its corresponding moment is defined to be the expected number of $\sigma$-equivariant surjections from $G$ to $H$, where $G$ is the $p$-tower group of $K$ as $K$ ranges through imaginary quadratic fields.

Theorem (B-Wood)

For every $H$ the corresponding moment is 1. Moreover this property characterizes the measure.

Theorem (B-Wood)

A function field version of this holds.

Indeed, moments for any group $H$ of odd order come out as 1.
Open Question 1

By Ledermann-Neumann, there are finitely many finite groups $G$ with a given $\text{Aut}(G)$.

**Question 1** Are there finitely many pairs $(G, \sigma)$, where $G$ is a finite group with GI-automorphism $\sigma$ (up to conjugation), with a given $\text{Aut}_\sigma(G)$?

This would be helpful in understanding $\text{Meas}^{BBH}$. 
Infinite Groups with Nonzero Measure?

Compare with the following situation.

Let $F$ be free pro-$2$ on 3 generators. Let $r \in \Phi(F)$.

Varying $r$ yields 3-generator 1-relator pro-$2$ groups.

Something surprising happens. For proportion $21/64$ of the choices for $r$, the group presented is the same group \( < x, y, z \mid x^y = x^3 z^2 > \).

Question 1 addresses whether any infinite Schur $\sigma$-group has nonzero measure. Note that Fontaine-Mazur says that no infinite quotient of a $p$-tower group is analytic.
Discussion of Open Question 1

It is easy to see that 2 choices of \((G, \sigma)\) yield \(\text{Aut}_\sigma(G)\) of order 1. It is a small exercise to see that 4 choices of \((G, \sigma)\) yield \(\text{Aut}_\sigma(G)\) of order 2.

There are apparently 13, 8, 47, 3, 42 choices of \((G, \sigma)\) with \(\text{Aut}_\sigma(G)\) of order 4, 6, 8, 10, 12 respectively.

Could \(\text{Aut}_\sigma(G)\) be a Klein 4-group infinitely often? For instance, there are infinitely many 2-groups \(A\) of maximal class possessing an involution \(\sigma\) with centralizer a Klein 4-group. Note, however, that the only 2-group of maximal class that is an automorphism group, is the dihedral group of order 8.
The second result with Wood suggests the following question:

**Question 2** Does there exist a group $G$ of odd order with two nonconjugate GI-automorphisms?

I checked this for all groups of order $< 2000$ (no examples).

Note that if $G$ is a $p$-group, then since the kernel from $\text{Aut}(G)$ to $\text{Aut}(G/\Phi(G))$ is a $p$-group, any two GI-automorphisms of $G$ are conjugate by Schur-Zassenhaus ($p$ odd).
Open Question 3

Let $G$ be a $d$-generator $p$-group ($p$ odd).
Let $F$ be the free group on $d$ generators, $x_1, \ldots, x_d$.
Let $\tau$ be the GI-automorphism of $F$ given by $x_i \mapsto x_i^{-1}$.

There are finitely many normal subgroups of $F$ with quotient isomorphic to $G$ and the orbits of $\tau$ on this set have size 1 or 2.

We say that $G$ possesses the **Kernel Invariance Property** (KIP) if all the orbits have the same size.

**Question 3** Does KIP hold for every Schur $\sigma$-group and every Schur $\sigma$-quotient?
Discussion of Open Question 3

For a group $G/P_c(G)$, its pMultiplicatorRank minus NuclearRank is at most $r(G)$.

The smallest groups failing KIP are SmallGroup(243,i) for $i = 51, \ldots, 55$.

These have 3 generators and pMultiplicatorRank minus NuclearRank equal to 6.

If one is $G/P_c(G)$, then the corresponding $G$ has $d(G) = 3$ and $r(G) \geq 6$ and so cannot be a Schur $\sigma$-group.

I checked 200 Schur $\sigma$-groups of order up to $3^{19}$ and their Schur $\sigma$-quotients and found they all satisfied KIP.