Profinite groups and the fixed points of coprime automorphisms

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GROUPS, RINGS, AND THEIR AUTOMORPHISMS
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Congratulations to Evgeny!!
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Influence of the fixed-point subgroup

Let $A$ be a finite group acting on a finite group $G$. We denote by

$$C_G(A) = \{ x \in G \mid x^a = x \text{ for any } a \in A \}$$

the fixed-point subgroup. Many well-known results show that the structure of the centralizer $C_G(A)$ of $A$ has influence over the structure of $G$.

**Thompson’s Theorem**

Let $A$ be a finite group of prime order acting on a finite group $G$. If $C_G(A) = 1$, then $G$ is nilpotent.

**Higman’s Theorem**

If $G$ is a nilpotent group admitting an automorphism $\phi$ of prime order $q$ and such that $C_G(\phi) = 1$, then $G$ has nilpotency class bounded by some function $h(q)$ depending on $q$ alone.
Coprime action

The influence of $C_G(A)$ over the structure of $G$ is specially strong if $(|A|, |G|) = 1$, that is, the action of $A$ on $G$ is coprime.

If the action is coprime, then...

- $C_{G/N}(A) = C_G(A)N/N$, for any $A$-invariant normal subgroup $N$.
- $G = \langle C_G(B) \mid A/B \text{ is cyclic} \rangle$ whenever $A$ is a noncyclic abelian group

Let $A#$ denote the set of non-identity elements of $A$. 
**Influence on the nilpotency class of $G$**

The following result is due to J. Ward.

**Theorem (1971)**

Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a finite group $G$ and assume that $C_G(a)$ is nilpotent for each $a \in A^\#$. Then $G$ is nilpotent.

There are examples showing that the above result fails if the order of $A$ is $q^2$.

Later on in 2001 the situation of the previous theorem was studied in detail by P. Shumyatsky.

**Theorem**

Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a finite group $G$ and assume that $C_G(a)$ is nilpotent of class at most $c$ for each $a \in A^\#$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $q$ and $c$. 
Action on profinite groups

A profinite group $G$ is a topological group that is isomorphic to an inverse limit of finite groups.

In this realm all the usual concepts of group theory are interpreted topologically, so for example, by a subgroup of a profinite group we mean a closed subgroup and we say that a subgroup $H$ is generated by a set $S$ if it is topologically generated by $S$.

By an automorphism of a profinite group we always mean a continuous automorphism.

A group $A$ of automorphisms of a profinite group $G$ is said to be coprime if $A$ has finite order while $G$ is an inverse limit of finite groups whose orders are relatively prime to the order of $A$. 
Influence of the fixed-point subgroup in profinite groups: from finite to profinite

Ward’s Theorem. Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a finite group $G$ and assume that $C_G(a)$ is nilpotent for each $a \in A^\#$. Then $G$ is nilpotent.

Using the routine inverse limit argument it is easy to deduce from the result above that

If $G$ is a profinite group admitting a coprime group of automorphisms $A$ of order $q^3$ such that $C_G(a)$ is pronilpotent for all $a \in A^\#$, then $G$ is pronilpotent.
In a similar way, from Shumyatsky’s Theorem. Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a finite group $G$ and assume that $C_G(a)$ is nilpotent of class at most $c$ for each $a \in A^\#$. Then $G$ is nilpotent and the class of $G$ is $\{q, c\}$-bounded.

It is possible to deduce that if $G$ is a profinite group admitting a coprime group of automorphisms $A$ of order $q^3$ such that $C_G(a)$ is nilpotent for all $a \in A^\#$, then $G$ is nilpotent.
Influence of the fixed-point subgroup in profinite groups

Note that certain results on fixed points in profinite groups cannot be deduced from corresponding results on finite groups.

Theorem (A. and Shumyatsky, 2016)

Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a profinite group $G$ and assume that $C_G(a)$ is locally nilpotent for each $a \in A^\#$. Then $G$ is locally nilpotent.

Recall that a group is locally nilpotent if every finitely generated subgroup is nilpotent.

The previous result looks similar to Ward’s and Shumyatsky’s Theorems but it cannot be deduced directly from those results. We need to use different techniques.
Engel groups

A group $G$ is an Engel group if for every $x, g \in G$,

$$[x, g, g, \ldots, g] = 1,$$

where $g$ is repeated sufficiently many times depending on $x$ and $g$.

If the number of times that $g$ is repeated in the commutator is $n$ and it does not depend on $x$ and $g$, then we say that $G$ is an $n$-Engel group.

Any locally nilpotent group is an Engel group.
Engel profinite groups

J. Wilson and E. Zelmanov proved the converse for profinite groups:

**Theorem (Wilson and Zelmanov, 1992)**
Any Engel profinite group is locally nilpotent.

If $G$ is an $n$-Engel group even more can be said

**Theorem (Burns and Medvedev, 1998)**
Let $G$ be an $n$-Engel profinite group. Then $G$ has a normal subgroup $N$ which is nilpotent of nilpotency class $c(n)$ and such that $G/N$ is of finite exponent $e(n)$.

Note that the nilpotency class $c(n)$ and the exponent $e(n)$ exclusively depends on the Engel class $n$. 
In view of this we can reformulate our result

Theorem

Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a profinite group $G$ and assume that $C_G(a)$ is Engel for each $a \in A^\#$. Then $G$ is Engel.

Zelmanov’s techniques created in the solution of the Restricted Burnside problem play an important role in our proof. Moreover some ideas that we are going to use in the proof are related to techniques developed to prove the following results.
Influence on the exponent of $G$

**Theorem (Khukhro and Shumyatsky, 1999)**

Let $q$ be a prime, $m$ a positive integer and $A$ an elementary abelian group of order $q^2$. Suppose that $A$ acts as a coprime group of automorphisms on a finite group $G$ and assume that $C_G(a)$ has exponent dividing $m$ for each $a \in A\#$. Then the exponent of $G$ is $\{m, q\}$-bounded.

Another quantitative result of similar nature is the following

**Theorem (Guralnick and Shumyatsky, 2001)**

Let $q$ be a prime, $m$ a positive integer and $A$ an elementary abelian group of order $q^3$. Assume that $A$ acts as a coprime group of automorphisms on a finite group $G$ and that $C_G(a)$ has derived group of exponent dividing $m$ for each $a \in A\#$. Then the exponent of $G'$ is $\{m, q\}$-bounded.

Note that the proof of the above result depends on the classification of finite simple groups and the assumption $|A| = q^3$ is essential, since the result fails if $|A| = q^2$. 
Idea of the proof

Recall that $q$ is a prime and $A$ is an elementary abelian $q$-group of order at least $q^3$. Assume that $A$ acts coprimely on a profinite group $G$ and assume that $C_G(a)$ is locally nilpotent for each $a \in A^\#$. We want to show that $G$ is locally nilpotent as well.

In view of the profinite version of Ward’s Theorem the group $G$ is pronilpotent and therefore $G$ is the Cartesian product of its Sylow subgroups.

We will need the following notation:

- For a profinite group $G$ we denote by $\pi(G)$ the set of prime divisors of the orders of finite continuous homomorphic images of $G$.
- If $m$ is an integer, we denote by $\pi(m)$ the set of prime divisors of $m$.
- If $\pi$ is a set of primes, we denote by $O_{\pi}(G)$ the maximal normal $\pi$-subgroup of $G$ and by $O_{\pi'}(G)$ the maximal normal $\pi'$-subgroup.
Idea of the proof

Proposition

For any locally nilpotent profinite group $K$ there exist a positive integer $n$, elements $g_1, g_2 \in K$ and an open subgroup $H \leq K$ such that the law $[x, ny] \equiv 1$ is satisfied on the cosets $g_1H, g_2H$, that is $[g_1h_1, ng_2h_2] = 1$ for all $h_1, h_2 \in H$.

Choose $a \in A^\#$. Since $C_G(a)$ is locally nilpotent, from the proposition above it follows that $C_G(a)$ contains an open subgroup $H$ and elements $u, v$ such that for some $n$ the law $[x, ny] \equiv 1$ is satisfied on the cosets $uH, vH$.

Let $[C_G(a) : H] = m$ and let $\pi_1(a) = \pi(m)$. Put $T = O_{\pi_1'}(C_G(a))$.

Since $T$ is isomorphic to the image of $H$ in $C_G(a)/O_{\pi_1}(C_G(a))$, it is easy to see that $T$ satisfies the law $[x, ny] \equiv 1$, that is, $T$ is $n$-Engel.

By the result of Burns and Medvedev the subgroup $T$ has a nilpotent normal subgroup $U$ such that $T/U$ has finite exponent, say $e$. Set $\pi_2(a) = \pi(e)$. 

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Idea of the proof

Note that the finite sets $\pi_1(a)$ and $\pi_2(a)$ depend on the choice of $a \in A^\#$. Let

$$\pi_a = \pi_1(a) \cup \pi_2(a)$$

and set

$$\pi = \bigcup_{a \in A^\#} \pi_a \text{ and } K = O_{\pi'}(G).$$

The choice of the finite set $\pi$ guarantees that $C_K(a)$ is nilpotent for every $a \in A^\#$. By the profinite version of Shumyatsky’s result, the subgroup $K$ is nilpotent.

We have

$$G = P_1 \times P_2 \times \cdots \times P_r \times K,$$

where $p_1, p_2, \ldots, p_r$ are the finitely many primes in $\pi$ and $P_1, P_2, \ldots, P_r$ are the corresponding Sylow subgroups of $G$.

For our purpose it is sufficient to show that each subgroup $P_i$ is locally nilpotent.
Idea of the proof

From now on w.l.o.g. we assume that $G$ is a pro-$p$ group for some prime $p$.

Since every finite subset of $G$ is contained in a finitely generated $A$-invariant subgroup we can also assume that $G$ is finitely generated.

We denote by $D_j = D_j(G)$ the terms of the $p$-dimension central series of $G$. The $D_j$ form a filtration and we can consider the Lie ring $L(G) = \bigoplus D_j / D_{j+1}$ that actually can be viewed as Lie algebra over $\mathbb{F}_p$.

Consider $L_p(G)$, the subalgebra generated by the first homogeneous component $D_1 / D_2$ in $L(G)$.

Since any automorphism of $G$ induces an automorphisms of $L(G)$, in particular the group $A$ naturally acts on $L_p(G)$.

By applying a combination of results due to Wilson, Zelmanov, Khukhro and Shumyatsky and the Lie-theoretical techniques created by Zelmanov in the solution of the RBP we can show that $L_p(G)$ is nilpotent.
Idea of the proof

Lazard’s Theorem, 1965
A finitely generated pro-$p$-group $G$ is $p$-adic analytic if and only if $L_p(G)$ is nilpotent.

Thus $G$ is $p$-adic analytic and we are in the position to apply

Theorem (Lubotzky and Mann, 1987)
A finitely generated pro-$p$ group $G$ is $p$-adic analytic if and only if it is of finite rank, that is, if all closed subgroups of $G$ are finitely generated.

Thus each centralizer $C_G(a)$ is finitely generated for every $a \in A^#$ and it follows that $C_G(a)$ are nilpotent.

Finally applying the profinite version of Shumyatsky’s result we get that $G$ is nilpotent. This concludes our proof.
Thank you!